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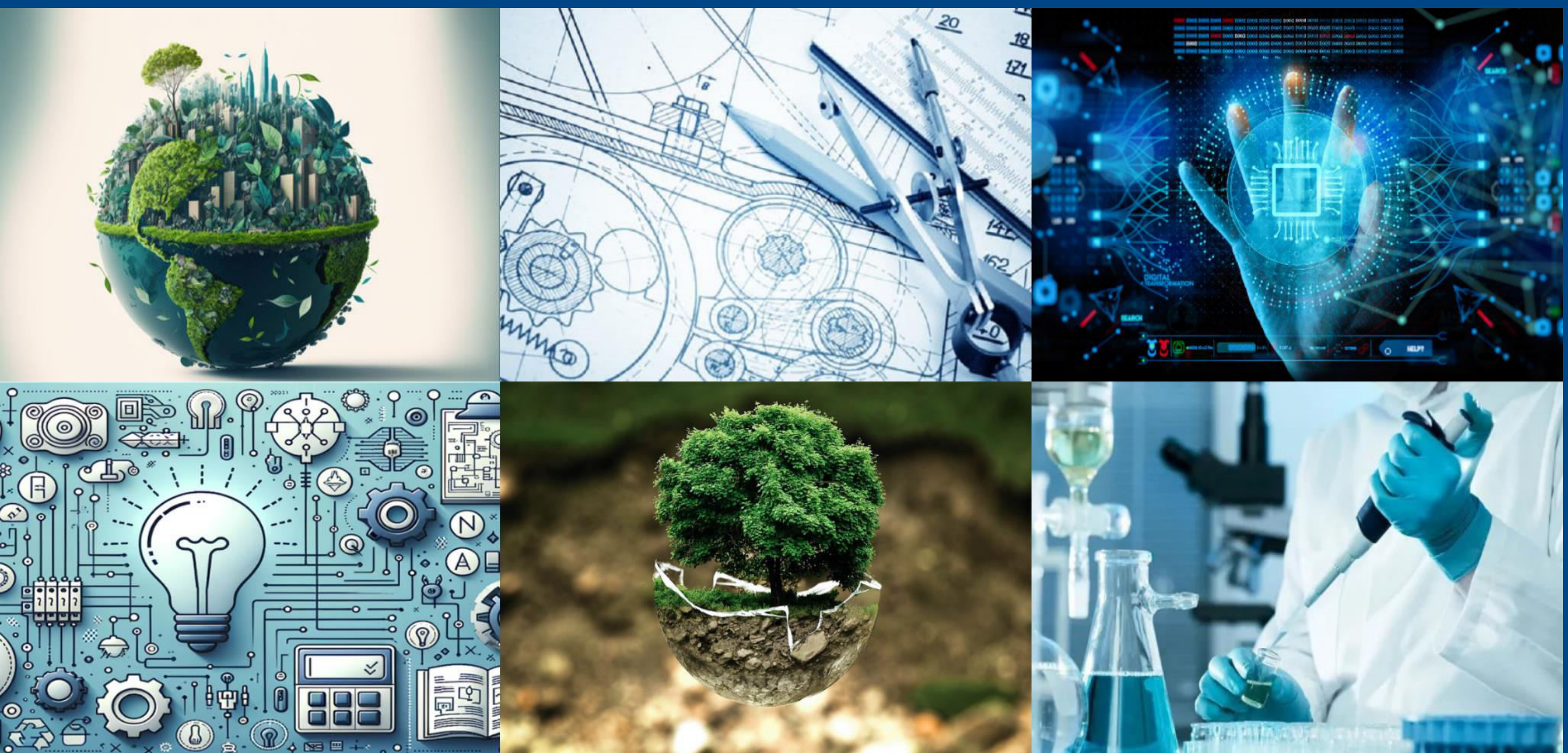

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Unique Extension of the Kannan-Chatterjea Theorem under Generalized Contractions

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ABSTRACT: This study used upper semi-continuous mappings to demonstrate the existence and uniqueness of common fixed-point theorems for entire metric spaces. Our findings extend to existing literature, Banach's fixed-point theorem, Kannan's fixed-point theorem, and Chatterjea's fixed-point theorem.

KEYWORDS: contraction maps, fixed point, normed space. **MSC 2000:** 47H09, 47H10**,** 46B40**.**

I. INTRODUCTION

The Study of nonlinear analysis is an important aspect of mathematics, especially of fixed point theory has emerged as a vital component in various branches of mathematics, particularly in analysis, topology, and applied sciences. At the heart of this theory lies the concept of standard spaces, structured settings that allow researchers to rigorously define and explore fixed points.

The development and refinement of such spaces, including metric, normed, and Banach spaces, play a crucial role in extending the applicability of fixed point results to more complex and realistic models, see([3, 5, 10, 14]).

Understanding the evolution of these spaces provides essential insight into the modern applied study of fixed points, enabling the formulation of more generalized and powerful theorems. This, in turn, enhances the ability to solve real-world problems, from engineering systems and economic models to differential equations and optimization problems. Thus, the continued development and analysis of standard spaces are not only of theoretical interest but also serve as a foundation for significant advancements in applied mathematics, see ([4, 7, 11])

In this paper, we tried to give two extensions with the help of the Kannan fixed point theorem and the Chatterjea fixed point theorem combined. To prove the extensions of the Banach Fixed Point Theorem, we used the technique of the successive iteration method for proving that the space is a complete metric space. We have proved the existence and uniqueness of common fixed-point theorems with the help of upper semi-continuous self-mappings on a complete metric space$(\mathbb{U}, \sigma)$.

Our results generalize Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem in the existing literature.

The presence of solutions for a number of nonlinear problems that have previously come up in the biological, physical, and social sciences, among other scientific fields, has been resolved since the Banach fixed point theorem [8] was first proposed in 1922.

The topic of whether a map of non-contractive type has a fixed point remained open to scholars after the Banach. Additionally, Kannan provided the following theorem in 1968 as the positive response in the situation of entire metric space.

Kannan's fixed point theorem [17] was proved using Banach's fixed point theorem. Chatterjea fixed-point theorem related to the Kannan was given [9].

Several authors have established the generalisation of the Banach fixed point theorem, the Kannan fixed point theorem, and the Chatterjea fixed point theorem [15,16].

II. METHODOLOGY

The main result of this research paper includes a comparative analysis of the existing results in the literature on fixed-point theory. To prove our results, which are generalizations of various fixed-point theorems such as Banach's Fixed Point Theorem, Kannan's Fixed Point Theorem, and Chatterjea's Fixed Point Theorem, and their various extensions [18], we have used the following results with some preliminaries.

Definition 2.1[2] Let X be a non-empty set and consider the function $\sigma: \mathcal{U} \times \mathcal{U} \to [0, \infty)$ which satisfies the following conditions, for all $u, v, w \in \mathcal{U}$

$(\sigma_1): \sigma(u, u) = 0$
$(\sigma_2): \sigma(u, v) = \sigma(v, u) \Rightarrow u = v$
$(\sigma_3): \sigma(u, v) = \sigma(v, u)$
$(\sigma_4): \sigma(u, v) \leq \sigma(u, w) + \sigma(w, v)$

Then σ is called metric on $\mathcal{U}$ and $(\mathcal{U}, \sigma)$ is called a metric space.

Definition 2.2[2] A sequence $\{u_n\}$ in a metric space $(\mathcal{U}, \sigma)$ is said to be convergent to w, if $\lim_{n\to\infty} \sigma(u_n, w) = 0 = \lim_{n\to\infty} \sigma(w, u_n)$. Here w is called limit point of a sequence $\{u_n\}$.

Definition 2.3[1] A sequence $\{u_n\}$. in a metric space $(\mathcal{U}, \sigma)$ is said to be Cauchy sequence if for a given $\epsilon > 0$, there exist a $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0, \sigma(u_n, u_m) < \epsilon$.

Definition 2.4[1] A metric space $(\mathcal{U}, \sigma)$ is said to be complete, if every Cauchy sequence in $\mathcal{U}$ is convergent to a point in $\mathcal{U}$.

Definition 2.4[6] Let $(\mathcal{U}, \sigma)$ be a metric space and let $\xi: \mathcal{U} \to \mathcal{U}$ be a mapping. Then a point $u \in \mathcal{U}$ is a fixed point of f if $\xi u = u$.

Definition 2.6[13] A function $\tau: \mathbb{R} \to [0, \infty)$ is said to be an upper Semi-Continuous from right if for any sequence $\{u_n\}$ converging to u as $u \geq 0$, then $\lim_{n\to\infty} \sup \tau(u_n) \leq \tau(u)$.

Theorem 2.7 (Banach) [8] Let $(\mathcal{U}, \sigma)$ be a complete metric space and $\xi: \mathcal{U} \to \mathcal{U}$ be a contraction, i.e., ξ satisfies, $d(\xi u, \xi v) \leq \alpha d(u, v)$, for all $u, v \in \mathcal{U}$ and a fixed constant $\alpha < 1$. Then there exists a unique fixed point of ξ in $\mathcal{U}$.
Theorem 2.8(Kannan) [17] Let $\xi: \mathcal{U} \to \mathcal{U}$, where $(\mathcal{U}, \sigma)$ is a complete metric space and ξ satisfies the condition

$$\sigma(\xi u, \xi v) \leq \beta[\sigma(u, \xi u) + \sigma(v, \xi v)]$$

where $0 \leq \beta < \frac{1}{2}$ and $u, v \in \mathcal{U}$. Then ξ has a unique fixed point in $\mathcal{U}$.

Theorem 2.9 (Chatterjea) [9] Let $(\mathcal{U}, \sigma)$ be a complete metric space. Let ξ be a Chatterjea mapping on $\mathcal{U}$, i.e., there exists $r \in \left[0, \frac{1}{2}\right)$ satisfying for all $u, y \in \mathcal{U}$

$$\sigma(\xi u, \xi v) \leq r[\, \sigma(u, \xi v) + \sigma(v, \xi u)\,].$$

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Then ξ has a unique fixed point.

Theorem 2.10[12] Let $(\mathcal{U}, \sigma)$ be a complete metric space and $\xi: \mathcal{U} \to \mathcal{U}$ be a mapping satisfying the condition

$$\sigma(\xi u, \xi v) \leq a_1[\sigma(x, \xi x) + \sigma(v, \xi v)] + a_2[\sigma(u, \xi v) + \sigma(v, \xi u)]$$

for all $u, v \in \mathcal{U}$, $0 < a_1,\ a_2 < \frac{1}{2}$ and $a_1 + a_2 < \frac{1}{2}$. Then f has unique fixed point in $\mathcal{U}$.
The proved theorem in this paper is a new one and is studied by combining the use of existing results of the Kannan fixed point theorem and the Chatterjea fixed point theorem.

III. MAIN RESULT

Theorem 3.1. Assume that $(\mathcal{U}, \sigma)$ is a complete metric space and a self-mapping $\xi: \mathcal{U} \to \mathcal{U}$ satisfies for all $u, v \in \mathcal{U}$

$$\sigma(\xi u, \xi v) \leq a_1\tau(\sigma(u, v)) + a_2\tau[\sigma(u, \xi u) + \sigma(v, \xi v)] + a_3\tau[\sigma(u, \xi v) + \sigma(v, \xi u)] + a_4\tau\left[\frac{\sigma(u, \xi u) d(v, \xi v)}{\sigma(u, v)}\right] \quad (3.1)$$

where $\tau: [-\infty, \infty] \to [0, \infty)$ is upper semi continuous from the right and satisfies
$0 \leq \tau(r) < r, \forall r > 0, \tau(0) = 0$ with $0 < a_1, a_2, a_3, a_4 < 1$ and $0 < a_1 + 2a_2 + 2a_3 + a_4 < 1$.
Then there is a unique $w \in \mathcal{U}$, such that $\xi(w) = w$.
Proof. Let $u_0 \in X$ be an arbitrary but a fixed element. Define a sequence of iterates $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{U}$ by

$$u_1 = \xi u_0,\ u_2 = \xi^2 u_0,\ x_3 = \xi^3 u_0, \ldots, u_n = \xi u_{n-1} = \xi^n u_0$$

By the condition (3.1) on ξ, we get

$$\begin{aligned}\sigma(u_n, u_{n+1}) &= \sigma(\xi u_{n-1}, \xi u_n)\\ &\leq a_1\tau\big(\sigma(u_{n-1}, u_n)\big) + a_2\tau[\sigma(u_{n-1}, \xi u_{n-1}) + \sigma(u_n, \xi u_n)] + a_3\tau[\sigma(u_{n-1}, \xi u_n) + \sigma(u_n, \xi u_{n-1})] + a_4\tau\left[\frac{\sigma(u_{n-1}, \xi u_{n-1})\sigma(u_n, \xi u_n)}{\sigma(u_{n-1}, u_n)}\right]\\ &\leq a_1\tau\big(\sigma(u_{n-1}, u_n)\big) + a_2\tau[\sigma(u_{n-1}, u_n) + \sigma(u_n, u_{n+1})] + a_3\tau[\sigma(u_{n-1}, u_{n+1}) + \sigma(u_n, u_n)] + a_4\tau\left[\frac{\sigma(u_{n-1}, u_n)\sigma(u_n, u_{n+1})}{\sigma(u_{n-1}, u_n)}\right]\end{aligned}$$

$$\sigma(u_n, u_{n+1}) < a_1\sigma(u_{n-1}, u_n) + a_2\sigma(u_{n-1}, u_n) + a_2\sigma(u_n, u_{n+1}) + a_3\sigma(u_{n-1}, u_n) + a_3\sigma(u_n, u_{n+1}) + a_4\sigma(u_n, u_{n+1})$$

$$[1 - (a_2 + a_3 + a_4)]\sigma(u_n, u_{n+1}) < (a_1 + a_2 + a_3)\sigma(u_{n-1}, u_n)$$

$$\sigma(u_n, u_{n+1}) < \frac{a_1 + a_2 + a_3}{1 - (a_2 + a_3 + a_4)}\sigma(u_{n-1}, u_n)$$

Then, $\sigma(u_n, u_{n+1}) < \mathcal{F}\, \sigma(u_{n-1}, u_n)$, where $\left(\mathcal{F} = \frac{a_1 + a_2 + a_3}{1 - (a_2 + a_3 + a_4)}\right)$

Here, $0 < \mathcal{F} < 1$, since $0 < a_1 + 2a_2 + 2a_3 + a_4 < 1$, $a_1, a_2, a_3, a_4 > 0$.

Continuing in this way, we get $\sigma(u_n, u_{n+1}) < \mathcal{F}^n\, \sigma(u_0, u_1)$.
Taking limit as $n \to \infty$, we get $\sigma(u_n, u_{n+1}) \to 0$, since $(0 < \mathcal{F} < 1)$.
Therefore, $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{U}$. As $\mathcal{U}$ is a complete metric space, there exist $w \in \mathcal{U}$ such that $\lim_{n\to\infty} u_n = w$.

We shall show that w is a fixed point of ξ.
As ξ is a continuous function, so we have

$$w = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \xi u_{n-1} = \xi\left(\lim_{n\to\infty} u_{n-1}\right) = \xi w$$

Therefore, $\xi w = w$ and hence w is a fixed point of ξ.
Now, we will show that w is a unique fixed point of ξ. Let $w_1 \in \mathcal{U}$ be another fixed point of ξ such that $w \neq w_1$. Again, by the (3.1), we have

$$\begin{aligned}
\sigma(w, w_1) &= \sigma(\xi w, \xi w_1) \\
&\leq a_1\tau(\sigma(w, w_1)) + a_2\tau[\sigma(w, \xi w) + \sigma(w_1, \xi w_1)] + a_3\tau[\sigma(w, \xi w_1) + \sigma(w_1, \xi w)] + a_4\tau\left[\frac{\sigma(w,\xi w)\sigma(w,\xi w_1)}{\sigma(w,w_1)}\right] \\
&\leq a_1\tau(\sigma(w, w_1)) + a_2\tau[\sigma(w, w) + \sigma(w_1, w_1)] + a_3\tau[\sigma(w, w_1) + \sigma(w_1, w)] + a_4\tau\left[\frac{\sigma(w,w)\sigma(w_1,w_1)}{\sigma(w,w_1)}\right] \\
&\leq a_1\tau(\sigma(w, w_1)) + a_3\tau[\sigma(w, w_1) + \sigma(w_1, w)] \\
&\leq a_1\tau(\sigma(w, w_1)) + a_3\tau[2\sigma(w, w_1)]
\end{aligned}$$

$$\sigma(w, w_1) < (a_1 + 2a_3)\sigma(w, w_1)$$

This is possible only if $\sigma(w, w_1) = 0 \quad \Rightarrow w = w_1$ which is a contradiction.
Therefore, $w \in \mathcal{U}$ is a unique element such that $\xi(w) = w$.

IV. CONCLUSION

In this research article, we proved the existence and uniqueness of common fixed point theorems for complete metric space with the help of upper semi continuous mapping. Our results generalizes Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem in existing literature.

V. OPEN PROBLEMS

- Can the current generalized contractive conditions be extended to more abstract settings such as G-metric spaces and partial metric spaces? Investigating the existence and uniqueness of fixed points in such spaces remains an open and promising area.

- Extend the generalized Kannan-Chatterjea types to study common and coinciding fixed points, which are particularly relevant in solving systems of equations and in multi-agent systems.

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