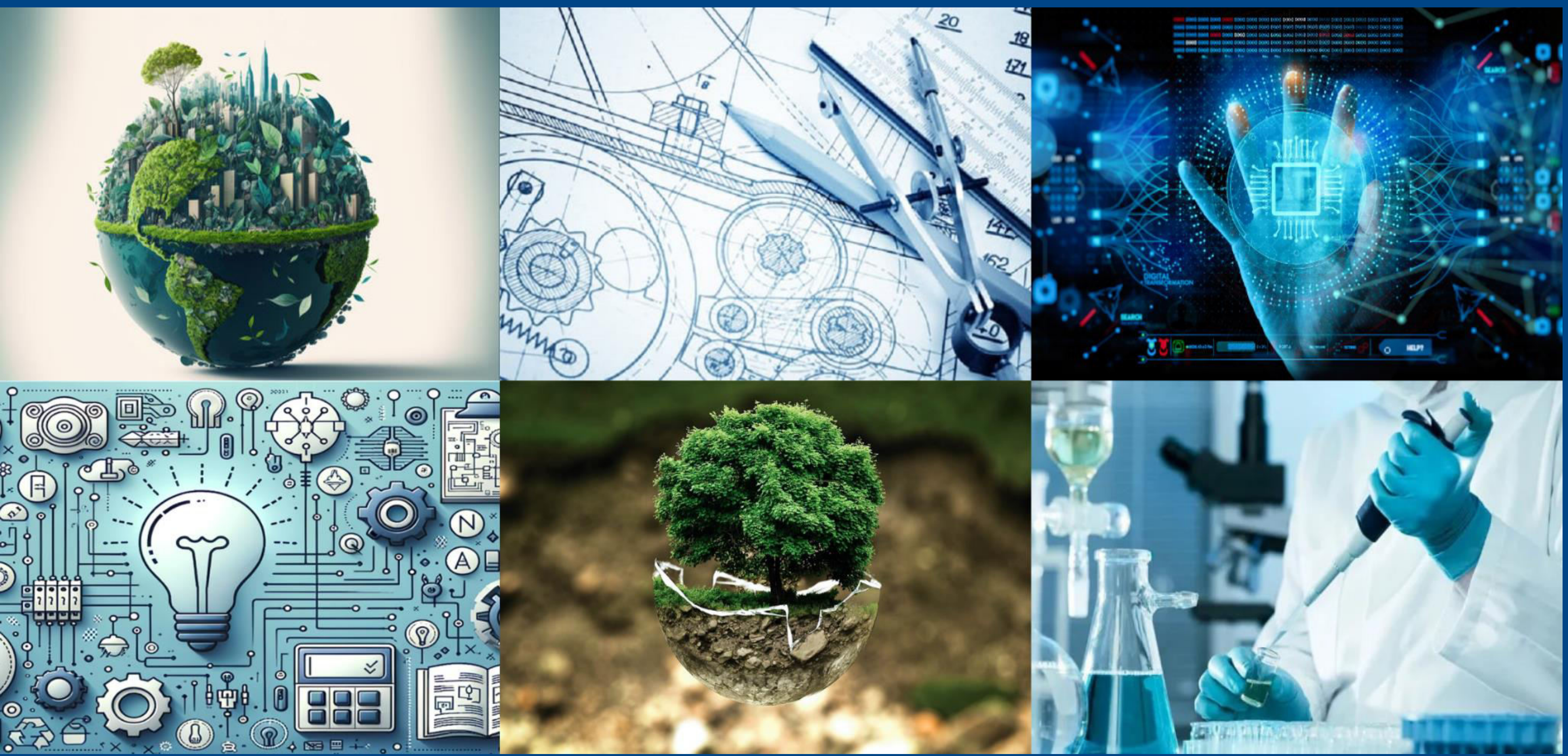
ISSN: 2582-7219

International Journal of Multidisciplinary Research in Science, Engineering and Technology

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)



Impact Factor: 7.521

Volume 8, Issue 1, January 2025

© 2025 IJMRSET | Volume 8, Issue 1, January 2025| DOI: 10.15680/IJMRSET.2025.0801001

ISSN: 2582-7219 | www.ijmrset.com | Impact Factor: 7.521| ESTD Year: 2018|

International Journal of Multidisciplinary Research in Science, Engineering and Technology (IJMRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)

A Study of C^{hv}-Mixed Birecurrent Finsler Space in Cartan Sense

Adel M. Al- Qashbari [1], Alaa A. Abdallah[*2], Fatma A. Ahmed [3]

Department of Engineering, Faculty of Engineering and Computing, University of Science & Technology-Aden, Yemen[1]

Department of Mathematics, Faculty of Education, Aden University, Aden, Yemen [1, 3]

Department of Mathematics, Faculty of Education, Abyan University, Abyan, Yemen [2]

ABSTRACT: This paper investigates the properties of C^{hv}-mixed birecurrent Finsler spaces within the Cartan sense. By employing specific methods. In this paper, we introduce Finsler space F_n who's the torsion tensor C^i_{jk} satisfies the birecurrent conditions with respect to the first and second kind of covariant derivatives in Cartan sense. Also, we study the relationship between Cartan's first curvature tensor S^i_{jkh} and $(h)hv$-torsion tensor C^i_{jk} in sense of Cartan.

KEYWORDS: Birecurrent C^{hv}-mixed birecurrent space, hv-covariant derivative of mixed second order, Cartan's first curvature tensor S^i_{jkh}.

I. INTRODUCTION & PRELIMINARIES

Finsler geometry, a generalization of Riemannian geometry, provides a powerful framework for studying spaces with anisotropic metric properties. This research delves into the intricate realm of Finsler spaces, focusing on the concept of birecurrence and its various generalizations. Previous studies have explored different aspects of birecurrent Finsler spaces, including those involving specific types of torsion tensors (Abdallah et al., 2022), as well as the role of higher-order derivatives in analyzing curvature tensors (Al-Qashbari et al., 2024).

Furthermore, researchers have investigated the extensions and developments of generalized recurrent Finsler spaces (Al-Qashbari et al., 2024) and examined the properties of Weyl's and conformal curvature tensors within the context of Finsler geometry (Al-Qashbari and Halboup, 2024). In recent years, there has been a surge of interest in Mixed Birecurrent Finsler Spaces. However, the study of C^{hv} mixed birecurrent Finsler spaces within the Cartan sense remains relatively unexplored. The concepts of C^h-recurrent space and C^v-birecurrent space are introduced by (Matsumoto, 1971). The properties of C^h −recurrent and C^v −recurrent for torsion tensor field of the second order in Finsler space have been studied by (Misra and Lodhi, 2008). Further, various special forms of the (h)hv-torsion tensor C_{ijk} in recurrent and birecurrent space have been studied by [9, 10, 11]. The recurrence and birecurrence property have been discussed by [12, 13]. Recently, the generalized birecurrent Finsler space of mixed covariant derivatives in Cartan sense has been studied by [3].

An n −dimensional Finsler space F_n equipped with the metric function $F(x, y)$ satisfying the request conditions [1, 8]. The vectors y_i and y^i defined by

(1.1) $y_i = g_{ij}(x, y)y^j$.

The metric tensor g_{ij} and its associative g^{ij} are connected by

(1.2) $g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & if\ j = k , \\ 0 & if\ j \neq k . \end{cases}$

In view of (1.1) and (1.2), we have

ISSN: 2582-7219 | www.ijmrset.com | Impact Factor: 7.521| ESTD Year: 2018|

International Journal of Multidisciplinary Research in Science, Engineering and Technology (IJMRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)

(1.3) a) $\delta^i_j y_i = y_j$, b) $\delta^i_j\, y^j = y^i$, c) $\delta^i_j\, g_{ir} = g_{jr}$,
d) $\delta^i_j\, g^{jk} = g^{ik}$, e) $y_i\, y^i = F^2$ and f) $g^{ik} y_k = y^i$.

The $(\mathrm{h})\mathrm{hv}$-torsion tensor which is positively homogeneous of degree -1 in y^i and symmetric in all its indices introduced and defined by [1, 9]

$$C_{ijk} = \tfrac{1}{2}\dot{\partial}_i\, g_{jk} = \tfrac{1}{4}\dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2 .$$

And satisfies the following identities

(1.4) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$, b) $C^h_{jk}\, g_{ih} = C_{ijk}$, c) $C^h_{jk} = g^{ih} C_{ijk}$,
d) $C^h_{jk}\, y^j = C^h_{jk}\, y^k = 0$ and f) $y_h C^h_{jk} = 0$.

Also, the $(\mathrm{h})\mathrm{hv}$-torsion tensor C_{ijk} and the $(\mathrm{v})\mathrm{hv}$-torsion tensor C^h_{jk} are satisfying the following conitions

(1.5) a) $C^h_{jk}\, g^{jk} = C^h$, b) $C_{ijk}\, g^{jk} = C_i$, c) $C^h_{jh} = C_j$,
d) $\left(\dot{\partial}_i C^h_{jk}\right) y^k = -\, C^h_{ji}$ and e) $C_i\, y^i = 0$.

Cartan [6] deduced the covariant derivatives of an arbitrary vector field X^i with respect to x^k which given by

(1.6) $X^i\,\big|_k = \dot{\partial}_k X^i + X^r C^{\,i}_{rk}$

and

(1.7) $X^i_{|k} = \partial_k\, X^i - \left(\dot{\partial}_r\, X^i\right) G^{\,r}_{\,k} + X^r\, \Gamma^{*i}_{rk}$,

where the function Γ^{*i}_{rk} is defined by $\Gamma^{*i}_{rk} = \Gamma^{i}_{rk} - C^i_{mr}\, \Gamma^m_{sk}\, y^s$. The functions Γ^{*i}_{rk} and G^r_k are connected by $G^r_k = \Gamma^{*r}_{sk} y^s$, where $\partial_j \equiv \frac{\partial}{\partial x^j}$, $\dot{\partial}_j \equiv \frac{\partial}{\partial y^j}$, $G^i_j = \dot{\partial}_j G^i$.

The equations (1.6) and (1.7) give two kinds of covariant differentiations which are called v-covariant differentiation (Cartan's first kind covariant derivative) and h-covariant differentiation (Cartan's second kind covariant derivative), respectively. So $X^i|_k$ and $X^i_{|k}$ are v-covariant derivative and h-covariant derivative of the vector field X^i. Therefore, v-covariant derivative and h-covariant derivative of the vectors y^i , y_i and metric tensors g_{ij} and its associative g^{ij} are satisfied [8 ,10]

(1.8) a) $g_{ij}\,\big|_k = 0$, b) $g_{ij|k} = 0$, c) $g^{ij}\,\big|_k = 0$, d) $g^{ij}_{|k} = 0$,
e) $y^i_{|k} = 0$, f) $y^i\big|_k = \delta^i_k$, g) $y_{j|k} = 0$ and h) $y_j|_k = g_{jk}$.

The tensor S^i_{jkh} called Cartan's first curvature tensor defined by

(1.9) $S^i_{jkh} = C^i_{kr}\, C^r_{jh} - C^i_{rh} C^r_{jk}$.

The associate curvature tensor S_{jpkh} , Ricci tensor S_{jk} , deviation tensor S^i_j and curvature scalar S of the curvature tensor S^i_{jkh} are given by

(1.10) a) $S_{jpkh} = g_{ip} S^i_{jkh}$, b) $S^i_{jki} = S_{jk}$, c) $S^i_j = S_{jk} g^{ik}$ and d) $S_{jk}\, g^{jk} = S$.

ISSN: 2582-7219 | www.ijmrset.com | Impact Factor: 7.521| ESTD Year: 2018|

International Journal of Multidisciplinary Research in Science, Engineering and Technology (IJMRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)

II. A C^{hv} - MIXED BIRECURRENT SPACE

A Finsler space F_n is called C^h-recurrent and C^v-recurrent if the $(h)hv$-torsion tensor C^i_{jk} satisfies the following conditions [7]

(2.1) $C^i_{jk|l} = k_l C^i_{jk}$,

(2.2) $C^i_{jk}|_m = a_m C^i_{jk}$.

Supposedly the $(h)hv$-torsion tensor C^i_{jk} satisfies the equation

(2.3) $C^i_{jk|l} = \lambda_l C^i_{jk} + \mu_l(\delta^i_k y_j - \delta^i_j y_k)$, $C^i_{jk} \neq 0$

Taking v-covariant derivative for (2.3), with respect to x^m, we get

$$C^i_{jk|l}|_m = \lambda_l|_m C^i_{jk} + \lambda_l C^i_{jk}|_m + \mu_l|_m(\delta^i_k y_j - \delta^i_j y_k) + \mu_l(\delta^i_k y_j - \delta^i_j y_k)_{|_m} .$$

Using the equation (2.2), (1.8h) and (1.8a,c) in the above equation we get

$$C^i_{jk|l|m} = \lambda_{l|m} C^i_{jk} + \lambda_l (a_m C^i_{jk}) + \mu_{l|m}(\delta^i_k y_j - \delta^i_j y_k) + \mu_l(\delta^i_k g_{jm} - \delta^i_j g_{km}) .$$

Or

(2.4) $C^i_{jk|l|m} = a_{lm} C^i_{jk} + b_{lm}(\delta^i_k y_j - \delta^i_j y_k) + \mu_l(\delta^i_k g_{jm} - \delta^i_j g_{km})$, $C^i_{jk} \neq 0$,

where $|l$ denote the h-covariant differentiation and $|m$ denote the v-covariant differentiation, where $a_{lm} = \lambda_{l|m} + \lambda_l a_m$ and $b_{lm} = \mu_{l|m}$ are non-zero covariant tensors field of second order and μ_l is non-zero covariant vector field of first order.

Definition 2.1. A Finsler space F_n which (h)hv-torsion tensor C^i_{jk} satisfies the condition (2.4) will be called a C^{hv}-mixed birecurrent Finsler space of second order and denoted it by $C^{hv} - (M)BRF_n$ and the tensor is called a hv-mixed birecurrent tensor and denote it by $hv - (M)BR$.

Remark 2.1. Every $C^h - RF_n$ maybe $C^{hv} - (M)BRF_n$.

Transvecting the condition (2.4) by g_{ih}, using (1.4b), (1.3c), (1.8a) and (1.8b), we get

(2.5) $C_{jhk|l|m} = a_{lm} C_{jhk} + b_{lm}(g_{kh} y_j - g_{jh} y_k) + \mu_l(g_{kh} g_{jm} - g_{jh} g_{km})$.

Contracting the indices i and k in the condition (2.4), using (1.5c), (1.2), (1.3a) and (1.3c),we get

(2.6) $C_{j|l|m} = a_{lm} C_j + b_{lm}(n-1) y_j + \mu_l(n-1) g_{jm}$.

Transvecting (2. 4) by g^{jk}, using (1.5a), (1.3a), (1.3b), (1.3c),(1.2), (1.8a)and (1.8b), we get

(2. 7) $C^i_{|l|m} = a_{lm} C^i$.

© 2025 IJMRSET | Volume 8, Issue 1, January 2025| DOI: 10.15680/IJMRSET.2025.0801001

ISSN: 2582-7219 | www.ijmrset.com | Impact Factor: 7.521| ESTD Year: 2018|

International Journal of Multidisciplinary Research in Science, Engineering and Technology (IJMRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)

Thus, we conclude

Corollary 2.1. In $C^{hv}-(M)BRF_n$, the tensors C_{jhk} , C_j and C^i are non-vanishing.

Transvecting the condition (2.4) by y^j, we get

(2.8) $C^i_{jk|l}|_m y^j + C^i_{jk|l} y^j|_m = \left(a_{lm} C^i_{jk} + b_{lm}(\delta^i_k y_j - \delta^i_j y_k) + \mu_l(\delta^i_k g_{jm} - \delta^i_j g_{km}) \right) y^j$

Using the equations (2.4) and (2.3) in above equation, using the conditions (1.3b), (1.3e), (1.4c), (1.4d),(1.2), (1.8e) and (1.8f), we get

(2.9) $[a_{lm} C^i_{jk} + b_{lm}(\delta^i_k y_j - \delta^i_j y_k) + \mu_l(\delta^i_k g_{jm} - \delta^i_j g_{km})]\, y^j$

$[\lambda_l C^i_{jk} + \mu_l(\delta^i_k y_j - \delta^i_j y_k)]\delta^j_m = [a_{lm} C^i_{jk} + b_{lm}(\delta^i_k y_j - \delta^i_j y_k) + \mu_l(\delta^i_k g_{jm} - \delta^i_j g_{km})] y^j.$

Or

(2.10) $\lambda_l C^i_{mk} + \mu_l(\delta^i_k y_m - \delta^i_m y_k) = 0$.

Transvecting (2.10) by g^{mk}, using (1.5a), (1.3d) and (1.3f), we get

(2.11) $C^i = 0.$

This implies $\lambda_l = 0$ or $C^i = 0$.

Also, transvecting (2.10) by y^k, using (1.4d), (1.3b) and (1.3e), we get

(2.12) $\mu_l(y^i y_m - \delta^i_m F^2) = 0$

This implies $\mu_l = 0$ or $y^i y_m = \delta^i_m F^2$, where $F^2 \neq 0$.

Therefore, based on the findings, we draw the conclusion that

Theorem 2.1. In $C^{hv}-(M)BRF_n$, we have $C^i = 0$ and $y^i y_m = \delta^i_m F^2$ if the covariant vectors field λ_l and μ_l are vanishing, respectively.

The Cartan's first curvature tensor S^i_{jkh} satisfies the equation (1.9). Taking h-covariant derivative and v-covariant derivative for (1.9) with respect to x^l and x^m respectively, we get

(2.13) $S^i_{jkh|l|m} = (C^i_{kr} C^r_{jh} - C^i_{rh} C^r_{jk})_{|l|m}$, $S^i_{jkh} \neq 0$,

or

$$S^i_{jkh|l|m} = C^i_{kr|l|m} C^r_{jh} + C^i_{kr|l} C^r_{jh|m} + C^i_{kr|m} C^r_{jh|l} + C^i_{kr} C^r_{jh|l|m} - C^i_{rh|l|m} C^r_{jk} - C^i_{rh|l} C^r_{jk|m} - C^i_{rh|m} C^r_{jk|l} - C^i_{rh} C^r_{jk|l|m}.$$

Using conditions (2.2) and (2.3) in the above equation, we get

(2.14) $S^i_{jkh|l|m} = [\, a_{lm} C^i_{kr} + b_{lm}(\delta^i_r y_k - \delta^i_k y_r) + \mu_l(\delta^i_r g_{km} - \delta^i_k g_{rm})] C^r_{jh}$

ISSN: 2582-7219 | www.ijmrset.com | Impact Factor: 7.521| ESTD Year: 2018|

International Journal of Multidisciplinary Research in Science, Engineering and Technology (IJMRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)

$$+ [\lambda_l C^i_{kr} + \mu_l(\delta^i_r y_k - \delta^i_k y_r)][\lambda_m C^r_{jh}] + [\lambda_m C^i_{kr}][\lambda_l C^r_{jh} + \mu_l(\delta^r_h y_j - \delta^r_j y_h)]$$
$$+ C^i_{kr}[a_{lm} C^r_{jh} + b_{lm}(\delta^r_h y_j - \delta^r_j y_h) + \mu_l(\delta^r_h g_{jm} - \delta^r_j g_{hm})]$$
$$- [a_{lm} C^i_{rh} + b_{lm}(\delta^i_h y_r - \delta^i_r y_h) + \mu_l(\delta^i_h g_{rm} - \delta^i_r g_{hm})] C^r_{jk}$$
$$- [\lambda_l C^i_{rh} + \mu_l(\delta^i_h y_r - \delta^i_r y_h)][\lambda_m C^r_{jk}] - [\lambda_m C^i_{rh}][\lambda_l C^r_{jk} + \mu_l(\delta^r_k y_j - \delta^r_j y_k)]$$
$$- C^i_{rh}[a_{lm} C^r_{jk} + b_{lm}(\delta^r_k y_j - \delta^r_j y_k) + \mu_l(\delta^r_k g_{jm} - \delta^r_j g_{km})].$$

Above equation can be written as

(2.15) $S^i_{jkh|l|m} = c_{lm} S^i_{jkh} + (d_{lm} y_k + 2\mu_l g_{km} + \kappa_{lm}) C^i_{jh} - \mu_l(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}),$

where $c_{lm} = 2\alpha_{lm} + 2\lambda_l \lambda_m$, $d_{lm} = 2b_{lm} + \lambda_l \lambda_m$ and $\kappa_{lm} = \mu_l \lambda_m$.

Transvecting (2.15) by g_{ip} , using (1.10a),(1.4b),(1.3c) and (1.8a,b), we get

(2.16) $S_{jpkh|l|m} = c_{lm} S_{jpkh} + (d_{lm} y_k + 2\mu_l g_{km} + \kappa_{lm}) C_{jph} - \mu_l(g_{kp} C_{jhm} - g_{hp} C_{jkm})$.

Contracting the indices i and h in (2.15), using (1.10b), (1.5c), (1.2) and (1.4b), we get

(2.17) $S_{jk|l|m} = c_{lm} S_{jk} + (d_{lm} y_k + 2\mu_l g_{km} + \kappa_{lm}) C_j - \mu_l(1+n) C_{jkm}.$

Transvecting (2.17) by g^{ik}, using (1.10c), (1.3f), (1.2), (1.4c) and (1.8a,b), we get

(2.18) $S^i_{j|l|m} = c_{lm} S^i_j + (d_{lm} y^i + 2\mu_l \delta^i_m + \kappa_{lm} g^{ik}) C_j - \mu_l(1+n) C^i_{jm}.$

Transvecting (2.17) by g^{jk}, using (1.10d), (1.5e), (1.5a,b) and (1.8a,b), we get

(2.19) $S_{|l|m} = c_{lm} S + \kappa_{lm} C^k + \mu_l(1-n) C_m.$

Consequently, we deduce that

Theorem 2.2. In $C^{hv} - (M)BRF_n$, the Cartan's first curvature tensor S^i_{jkh} , associate tensor S_{jpkh}, S –Ricci tensor S_{jk} , deviation tensor S^i_j and curvature scalar S of curvature tensor S^i_{jkh} are given by the equations (2.15), (2.16), (2.17), (2.18) and (2.19), respectively.

III. APPLICATIONS IN MATHEMATICS AND PHYSICS

The study of Finsler spaces, particularly $C^{hv} - (M)BRF_n$, and associated tensors like the hv –mixed birecurrent tensor, has significant implications in several fields, both in pure mathematics and theoretical physics. The properties of these spaces are central to various branches of differential geometry, theoretical relativity, and field theory.

The concept of a $C^{hv} - (M)BRF_n$ is a generalization of Finsler spaces with a higher-order structure, influenced by torsion and curvature. This extends classical Riemannian geometry and allows the study of spaces with anisotropic behavior. The Cartan curvature tensor and its associated tensors, as derived in the research, provide a deeper understanding of the geometric properties of these spaces, such as their curvature and torsion characteristics.

The study of the hv –mixed birecurrent tensor and its properties has applications in exploring the curvature of manifolds under more complex conditions than those typically encountered in Riemannian geometry. These generalized curvature tensors can be employed to understand spaces with anisotropic metrics, which are useful in

ISSN: 2582-7219 | www.ijmrset.com | Impact Factor: 7.521| ESTD Year: 2018|

International Journal of Multidisciplinary Research in Science, Engineering and Technology (IJMRSET)

(A Monthly, Peer Reviewed, Refereed, Scholarly Indexed, Open Access Journal)

higher-dimensional geometry and complex manifold theory. The study of birecurrent spaces has applications in material science and optical physics, particularly in the study of birefringence in materials. The equations involving the hv-mixed birecurrent tensors provide a framework for studying nonlinear differential equations in Finsler spaces. These systems often arise in various applications where the equations governing the dynamics are not easily solvable in standard Riemannian geometry.

The introduction of torsion tensors in the framework of Finsler geometry plays a crucial role in understanding the global structure of manifolds, especially in spaces where the geometry cannot be captured using traditional Riemannian metrics. The equations governing the torsion and birecurrence properties can be used to analyze the nature of geodesics and curvature in such spaces.

IV. CONCLUSION

In conclusion, the study of -spaces and their associated tensors opens up new avenues for both pure and applied mathematics. In mathematics, it advances the field of differential geometry by generalizing curvature and torsion concepts. In physics, it enhances our understanding of the geometric foundations of gravitational theories, field theories, and quantum mechanics in complex spacetimes. These advancements are crucial for both theoretical exploration and practical applications in a variety of domains.

REFERENCES

1. Abdallah A. A., Navlekar A. A., Ghadle K. P. and Hardan B. (2022). Several forms of $h(hv)$-torsion tensor C_{jkh} generalize βP −birecurrent space, International Journal of Advanced Research in Science, Engineering and Technology, 9(7), (2022), 19505-19510.
2. Al-Qashbari A. M. and AL-ssallal F. A. (2024). Study of curvature tensor by using Berwald's and Cartan's higher-order derivatives in Finsler space, Technological Applied and Humanitarian Academic Journal, 1(1), 1-15.
3. Al-Qashbari A. M., Abdallah A. A. and Baleedi S. M. (2024). A Study on the extensions and developments of generalized BK-recurrent Finsler space, International Journal of Advances in Applied Mathematics and Mechanics, 12(1), 38-45.
4. Al-Qashbari A. M., Abdallah A. A. and Ahmed F. A. (2024). On generalized birecurrent Finsler space of mixed covariant derivatives in Cartan sense, International Journal of Research Publication and Reviews, 5(8), (2024), 2834-2840.
5. Al-Qashbari A. M. and Halboup A. H. (2024). Some identities Weyl's curvature tensor and conformal curvature tensor, University of Lahej Journal of Applied Sciences and Humanities, 1(1), 1-9.
6. Cartan É (1971). Les espaces de Finsler, Actualités, Paris, 79 (1934), 2nd edit.
7. Misra C. K. and Lodhi G. (2008). On C^h −recurrent and C^v −recurrent Finsler space of second order, Int.J. Contemp. math. Sciences, 3(17), 801- 810.
8. Rund H. (1950). The differential geometry of Finsler space, Spring −Verlag, Berlin Gottingen − Heidelberg.
9. Matsumoto M. (1971). On h − Isotropic and C^h − recurrent Finsler, J. Math. Kyoto Univ., 11, 1-9.
10. Pandey P. N., and Verma R. (1997). C^h − birecurrent Finsler space, second Conference of the International Academy of physical sciences, 13-14.
11. Saleem A. A. and Abdallah A. A. (2023). Certain identities of C^h in Finsler spaces, International Journal of Advanced Research in Science, Communication and Technology, 3(2), 620-622.
12. Saleem A. A. and Abdallah A. A. (2024). The recurrence property for the projective curvature tensor in Finsler space, International Advanced Research Journal in Science, Engineering and Technology, 11(5), 291-297.
13. Saleem A. A. and Abdallah A. A. (2024). On birecurrent Finsler space for projective curvature tensor, International Journal of Advance and Applied Research, 11(6), 95-99.

INTERNATIONAL STANDARD SERIAL NUMBER INDIA

NISCAIR

INNO SPACE

SJIF Scientific Journal Impact Factor

INTERNATIONAL JOURNAL OF MULTIDISCIPLINARY RESEARCH IN SCIENCE, ENGINEERING AND TECHNOLOGY

| Mobile No: +91-6381907438 | Whatsapp: +91-6381907438 | ijmrset@gmail.com |

www.ijmrset.com